

COMPLEX NUMBER

1. DEFINITION

A number of the form $a + ib$, where $a, b \in \mathbb{R}$ and $i = \sqrt{-1}$, is called a complex number and is denoted by 'Z'.

$$z = \boxed{a} + i\boxed{b}$$

\downarrow \downarrow
 $\text{Re}(z)$ $\text{Im}(z)$

1.1 Conjugate of a Complex Number

For a given complex number $z = a + ib$, its conjugate ' \bar{z} ' is defined as $\bar{z} = a - ib$

2. ALGEBRA OF COMPLEX NUMBERS

Let $z_1 = a + ib$ and $z_2 = c + id$ be two complex numbers where $a, b, c, d \in \mathbb{R}$ and $i = \sqrt{-1}$.

1. Addition :

$$\begin{aligned} z_1 + z_2 &= (a + bi) + (c + di) \\ &= (a + c) + (b + d) i \end{aligned}$$

2. Subtraction :

$$\begin{aligned} z_1 - z_2 &= (a + bi) - (c + di) \\ &= (a - c) + (b - d) i \end{aligned}$$

3. Multiplication :

$$\begin{aligned} z_1 \cdot z_2 &= (a + bi)(c + di) \\ &= a(c + di) + bi(c + di) \\ &= ac + adi + bci + bdi^2 \\ &= ac - bd + (ad + bc) i \end{aligned}$$

$$(\because i^2 = -1)$$

4. Division :

$$\begin{aligned} \frac{z_1}{z_2} &= \frac{a + bi}{c + di} = \frac{a + bi}{c + di} \cdot \frac{c - di}{c - di} \\ &= \left(\frac{ac + bd}{c^2 + d^2} \right) + \left(\frac{bc - ad}{c^2 + d^2} \right) i \end{aligned}$$

Note...

$$1. \quad a + ib = c + id$$

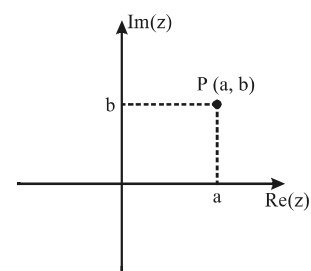
$$\Leftrightarrow a = c \ \& \ b = d$$

$$2. \quad i^{4k+r} = \begin{cases} 1; & r = 0 \\ i; & r = 1 \\ -1; & r = 2 \\ -i; & r = 3 \end{cases}$$

$$3. \quad \sqrt{\sqrt{}} \sqrt{a} = \sqrt{\sqrt{a}} \text{ only if atleast one of either a or b is non-negative.}$$

3. ARGAND PLANE

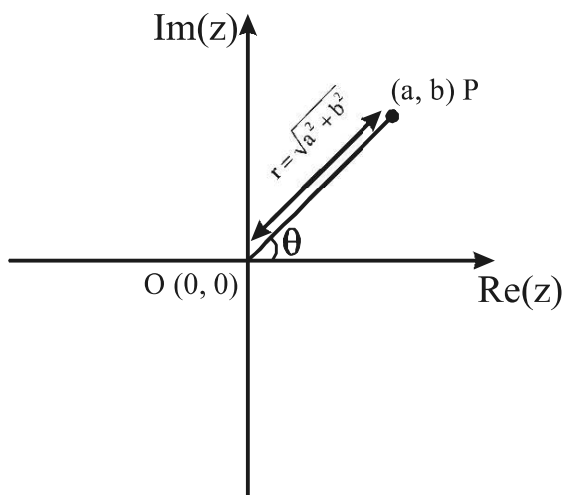
A complex number $z = a + ib$ can be represented by a unique point P (a, b) in the argand plane.



$Z = a + ib$ is represented by a point P (a, b)

3.1 Modulus and Argument of Complex Number

If $z = a + ib$ is a complex number



- (i) Distance of Z from origin is called as modulus of complex number Z.

It is denoted by $r = |z| = \sqrt{a^2 + b^2}$

- (ii) Here, θ i.e. angle made by OP with positive direction of real axis is called **argument of z**.

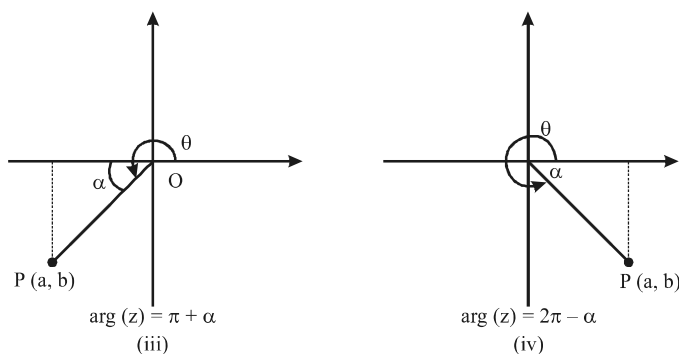
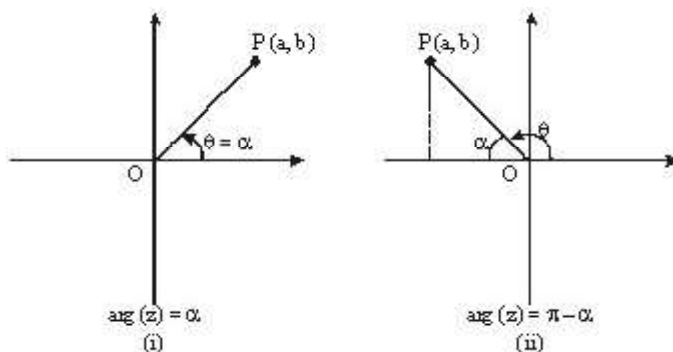
Note...

$z_1 > z_2$ or $z_1 < z_2$ has no meaning but $|z_1| > |z_2|$ or $|z_1| < |z_2|$ holds meaning.

3.2 Principal Argument

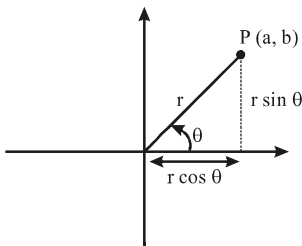
The argument ' θ ' of complex number $z = a + ib$ is called principal argument of z if $-\pi < \theta \leq \pi$.

Let $\tan \alpha = \left| \frac{b}{a} \right|$, and θ be the arg (z).



In (iii) and (iv) principal argument is given by $-\pi + \alpha$ and $-\alpha$ respectively.

4. POLAR FORM



$a = r \cos \theta$ & $b = r \sin \theta$;

where $r = |z|$ and $\theta = \arg(z)$

$\therefore z = a + ib$

$= r (\cos \theta + i \sin \theta)$



Note...

$Z = re^{i\theta}$ is known as Euler's form; where $r = |Z|$ & $\theta = \arg(Z)$

5. SOME IMPORTANT PROPERTIES

1. $\overline{(\overline{z})} = z$
2. $z + \overline{z} = 2 \operatorname{Re}(z)$
3. $z - \overline{z} = 2i \operatorname{Im}(z)$
4. $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$
5. $\overline{z_1 z_2} = \overline{z_1} \overline{z_2}$
6. $|z| = 0 \Rightarrow z = 0$
7. $z\overline{z} = |z|^2$
8. $|z_1 z_2| = |z_1| |z_2|$; $\left| \frac{z_1}{z_2} \right| = \frac{|z_1|}{|z_2|}$
9. $|\overline{z}| = |z| = |-z|$
10. $|z_1 \pm z_2|^2 = |z_1|^2 + |z_2|^2 \pm 2 \operatorname{Re}(z_1 \overline{z_2})$
11. $|z_1 + z_2| \leq |z_1| + |z_2|$ (Triangle Inequality)
12. $|z_1 - z_2| \geq ||z_1| - |z_2||$
13. $|az_1 - bz_2|^2 + |bz_1 + az_2|^2 = (a^2 + b^2)(|z_1|^2 + |z_2|^2)$

14. $\operatorname{amp}(z_1 \cdot z_2) = \operatorname{amp} z_1 + \operatorname{amp} z_2 + 2k\pi$; $k \in \mathbb{I}$

15. $\operatorname{amp} \left(\frac{y_0}{y_1} \right) = \operatorname{amp} z_1 - \operatorname{amp} z_2 + 2k\pi$; $k \in \mathbb{I}$

16. $\operatorname{amp}(z^n) = n \operatorname{amp}(z) + 2k\pi$; $k \in \mathbb{I}$

6. DE-MOIVRE'S THEOREM

Statement : $\cos n\theta + i \sin n\theta$ is the value or one of the values of $(\cos \theta + i \sin \theta)^n$ according as if 'n' is integer or a rational number. The theorem is very useful in determining the roots of any complex quantity

7. CUBE ROOT OF UNITY

Roots of the equation $x^3 = 1$ are called cube roots of unity.

$x^3 - 1 = 0$

$(x - 1)(x^2 + x + 1) = 0$

$x = 1$ or $x^2 + x + 1 = 0$

i.e $x = \frac{-1 + \sqrt{3}i}{2}$ or $x = \frac{-1 - \sqrt{3}i}{2}$

- (i) The cube roots of unity are $1, \frac{-1 + i\sqrt{3}}{2}, \frac{-1 - i\sqrt{3}}{2}$.
- (ii) $\omega^3 = 1$
- (iii) If w is one of the imaginary cube roots of unity then $1 + w + w^2 = 0$.
- (iv) In general $1 + w^r + w^{2r} = 0$; where $r \in \mathbb{I}$ but is not the multiple of 3.
- (v) In polar form the cube roots of unity are :
 $\cos 0 + i \sin 0$; $\cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3}$, $\cos \frac{4\pi}{3} + i \sin \frac{4\pi}{3}$
- (vi) The three cube roots of unity when plotted on the argand plane constitute the vertices of an equilateral triangle.
- (vii) The following factorisation should be remembered :

$a^3 - b^3 = (a - b)(a - \omega b)(a - \omega^2 b)$;

$x^2 + x + 1 = (x - \omega)(x - \omega^2)$;

$a^3 + b^3 = (a + b)(a + \omega b)(a + \omega^2 b)$;

$a^3 + b^3 + c^3 - 3abc = (a + b + c)(a + \omega b + \omega^2 c)(a + \omega^2 b + \omega c)$

8. 'n' nth ROOTS OF UNITY

Solution of equation $x^n = 1$ is given by

$$x = \cos \frac{2k\pi}{n} + i \sin \frac{2k\pi}{n} ; k = 0, 1, 2, \dots, n-1$$

$$= e^{i\left(\frac{2k\pi}{n}\right)} ; k = 0, 1, \dots, n-1$$



Note...

- We may take any n consecutive integral values of k to get 'n' nth roots of unity.
- Sum of 'n' nth roots of unity is zero, $n \in \mathbb{N}$
- The points represented by 'n' nth roots of unity are located at the vertices of regular polygon of n sides inscribed in a unit circle, centred at origin & one vertex being one +ve real axis.

Properties :

If $1, \alpha_1, \alpha_2, \alpha_3, \dots, \alpha_{n-1}$ are the n, nth root of unity then :

- (i) They are in G.P. with common ratio $e^{i(2\pi/n)}$
- (ii) $1^p + \alpha_0^p + \alpha_1^p + \dots + \alpha_{n-1}^p = \begin{cases} 0, & \text{if } p \neq kn \\ n, & \text{if } p = kn \end{cases}$ where $k \in \mathbb{Z}$
- (iii) $(1 - \alpha_1)(1 - \alpha_2) \dots (1 - \alpha_{n-1}) = n$
- (iv) $(1 + \alpha_1)(1 + \alpha_2) \dots (1 + \alpha_{n-1}) = \begin{cases} 0, & \text{if } n \text{ is even} \\ 1, & \text{if } n \text{ is odd} \end{cases}$
- (v) $1 \cdot \alpha_1 \cdot \alpha_2 \cdot \alpha_3 \dots \alpha_{n-1} = \begin{cases} -1, & \text{if } n \text{ is even} \\ 1, & \text{if } n \text{ is odd} \end{cases}$



Note...

- (i) $\cos \theta + \cos 2\theta + \cos 3\theta + \dots + \cos n\theta = \frac{\sin(n\theta/2)}{\sin(\theta/2)} \cos\left(\frac{n+1}{2}\theta\right)$
- (ii) $\sin \theta + \sin 2\theta + \sin 3\theta + \dots + \sin n\theta = \frac{\sin(n\theta/2)}{\sin(\theta/2)} \sin\left(\frac{n+1}{2}\theta\right)$

9. SQUARE ROOT OF COMPLEX NUMBER

Let $x + iy = \sqrt{a + ib}$, Squaring both sides, we get

$$(x + iy)^2 = a + ib$$

i.e. $x^2 - y^2 = a, 2xy = b$

Solving these equations, we get square roots of z.

10. LOCI IN COMPLEX PLANE

- (i) $|z - z_0| = a$ represents circumference of circle, centred at z_0 , radius a.
- (ii) $|z - z_0| < a$ represents interior of circle
- (iii) $|z - z_0| > a$ represents exterior of this circle.
- (iv) $|z - z_1| = |z - z_2|$ represents \perp bisector of segment with end points z_1 & z_2 .
- (v) $\left| \frac{z - z_1}{z - z_2} \right| = k$ represents : $\left\{ \begin{array}{l} \text{circle, } k \neq 1 \\ \perp \text{ bisector, } k = 1 \end{array} \right\}$
- (vi) $\arg(z) = \theta$ is a ray starting from origin (excluded) inclined at an $\angle \theta$ with real axis.
- (vii) Circle described on line segment joining z_1 & z_2 as diameter is :
 $(z - z_1)(\bar{z} - \bar{z}_2) + (z - z_2)(\bar{z} - \bar{z}_1) = 0$.
- (viii) Four pts. z_1, z_2, z_3, z_4 in anticlockwise order will be concyclic, if & only if

$$\theta = \arg\left(\frac{z_2 - z_4}{z_1 - z_4}\right) = \arg\left(\frac{z_2 - z_3}{z_1 - z_3}\right)$$

$$\Rightarrow \arg\left(\frac{z_2 - z_4}{z_1 - z_4}\right) - \arg\left(\frac{z_2 - z_3}{z_1 - z_3}\right) = 2n\pi ; (n \in \mathbb{I})$$

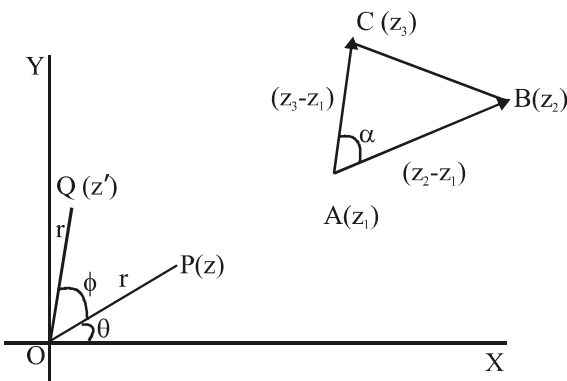
$$\Rightarrow \arg\left[\left(\frac{z_2 - z_4}{z_1 - z_4}\right)\left(\frac{z_1 - z_3}{z_2 - z_3}\right)\right] = 2n\pi$$

$$\Rightarrow \left(\frac{z_2 - z_4}{z_1 - z_4}\right) \times \left(\frac{z_1 - z_3}{z_2 - z_3}\right) \text{ is real \& positive.}$$

11. VECTORIAL REPRESENTATION OF A COMPLEX

Every complex number can be considered as if it is the position vector of that point. If the point P represents the complex number z then,

$$\vec{OP} = z \quad \& \quad |\vec{OP}| = |z|.$$



Note...

(i) If $\vec{OP} = z = r e^{i\theta}$ then $\vec{OQ} = z_1 = r e^{i(\theta + \phi)} = z \cdot e^{i\phi}$.

If \vec{OP} and \vec{OQ} are of unequal magnitude then

$$\hat{OQ} = \hat{OP} e^{i\phi}$$

(ii) If z_1, z_2, z_3 are three vertices of a triangle ABC described in the counter-clock wise sense, then

$$\frac{z_3 - z_2}{z_2 - z_1} = \frac{AC}{AB} (\cos \alpha + i \sin \alpha) = \frac{AC}{AB} \cdot e^{i\alpha} = \frac{|z_3 - z_2|}{|z_2 - z_1|} \cdot e^{i\alpha}$$

12. SOME IMPORTANT RESULTS

- (i) If z_1 and z_2 are two complex numbers, then the distance between z_1 and z_2 is $|z_2 - z_1|$.
- (ii) Segment Joining points A (z_1) and B(z_2) is divided by point P (z) in the ratio $m_1 : m_2$

then $z = \frac{m_1 z_2 + m_2 z_1}{m_1 + m_2}$, m_1 and m_2 are real.

(iii) The equation of the line joining z_1 and z_2 is given by

$$\begin{vmatrix} z & \bar{z} \\ z_1 & \bar{z}_1 \\ z_2 & \bar{z}_2 \end{vmatrix} = 0 \quad (\text{non parametric form})$$

Or

$$\frac{z - z_1}{\bar{z} - \bar{z}_1} = \frac{z - z_2}{\bar{z} - \bar{z}_2}$$

- (iv) $\bar{a}z + a\bar{z} + b = 0$ represents general form of line.
- (v) The general eqn. of circle is :
 $z\bar{z} + a\bar{z} + \bar{a}z + b = 0$ (where b is real no.).

Centre : $(-a)$ & radius $\sqrt{|a|^2 - b} = \sqrt{a\bar{a} - b}$.

(vi) Circle described on line segment joining z_1 & z_2 as diameter is :

$$(z - z_1)(\bar{z} - \bar{z}_2) + (z - z_2)(\bar{z} - \bar{z}_1) = 0.$$

(vii) Four pts. z_1, z_2, z_3, z_4 in anticlockwise order will be concyclic, if & only if

$$\theta = \arg\left(\frac{z_2 - z_4}{z_1 - z_4}\right) = \arg\left(\frac{z_2 - z_3}{z_1 - z_3}\right)$$

$$\Rightarrow \arg\left(\frac{z_2 - z_4}{z_1 - z_4}\right) - \arg\left(\frac{z_2 - z_3}{z_1 - z_3}\right) = 2n\pi; (n \in \mathbb{I})$$

$$\Rightarrow \arg\left[\left(\frac{z_2 - z_4}{z_1 - z_4}\right)\left(\frac{z_1 - z_3}{z_2 - z_3}\right)\right] = 2n\pi$$

$$\Rightarrow \left(\frac{z_2 - z_4}{z_1 - z_4}\right) \times \left(\frac{z_1 - z_3}{z_2 - z_3}\right) \text{ is real \& positive.}$$

(viii) If z_1, z_2, z_3 are the vertices of an equilateral triangle where z_0 is its circumcentre then

$$(a) \frac{1}{z_2 - z_3} + \frac{1}{z_3 - z_1} + \frac{1}{z_1 - z_2} = 0$$

$$(b) z_0^1 + z_1^1 + z_2^1 - z_1 z_2 - z_2 z_3 - z_3 z_1 = 0$$

$$(c) z_0^1 + z_1^1 + z_2^1 = 3 z_1^1$$

(ix) If A, B, C & D are four points representing the complex numbers z_1, z_2, z_3 & z_4 then

$$AB \parallel CD \text{ if } \frac{z_4 - z_3}{z_2 - z_1} \text{ is purely real ;}$$

$$AB \perp CD \text{ if } \frac{z_4 - z_3}{z_2 - z_1} \text{ is purely imaginary]}$$

(x) Two points P (z_1) and Q (z_2) lie on the same side or opposite side of the line $\bar{a}z + a\bar{z} + b$ accordingly as $\bar{a}z_1 + a\bar{z}_1 + b$ and $\bar{a}z_2 + a\bar{z}_2 + b$ have same sign or opposite sign.

Important Identities

$$(i) x^2 + x + 1 = (x - \omega)(x - \omega^2)$$

$$(ii) x^2 - x + 1 = (x + \omega)(x + \omega^2)$$

$$(iii) x^2 + xy + y^2 = (x - y\omega)(x - y\omega^2)$$

$$(iv) x^2 - xy + y^2 = (x + y\omega)(x + y\omega^2)$$

$$(v) x^2 + y^2 = (x + iy)(x - iy)$$

$$(vi) x^3 + y^3 = (x + y)(x + y\omega)(x + y\omega^2)$$

$$(vii) x^3 - y^3 = (x - y)(x - y\omega)(x - y\omega^2)$$

$$(viii) x^2 + y^2 + z^2 - xy - yz - zx = (x + y\omega + z\omega^2)(x + y\omega^2 + z\omega)$$

$$\text{or } (x\omega + y\omega^2 + z)(x\omega^2 + y\omega + z)$$

$$\text{or } (x\omega + y + z\omega^2)(x\omega^2 + y + z\omega).$$

$$(ix) x^3 + y^3 + z^3 - 3xyz = (x + y + z)(x + \omega y + \omega^2 z)(x + \omega^2 y + \omega z)$$

QUADRATIC EQUATION

1. QUADRATIC EXPRESSION

The general form of a quadratic expression in x is,

$$f(x) = ax^2 + bx + c, \text{ where } a, b, c \in \mathbb{R} \text{ \& } a \neq 0.$$

and general form of a quadratic equation in x is,

$$ax^2 + bx + c = 0, \text{ where } a, b, c \in \mathbb{R} \text{ \& } a \neq 0.$$

2. ROOTS OF QUADRATIC EQUATION

(a) **The solution of the quadratic equation,**

$$ax^2 + bx + c = 0 \text{ is given by } x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The expression $D = b^2 - 4ac$ is called the discriminant of the quadratic equation.

(b) **If α & β are the roots of the quadratic equation**

$$ax^2 + bx + c = 0, \text{ then ;}$$

$$(i) \alpha + \beta = -b/a \quad (ii) \alpha \beta = c/a$$

$$(iii) |\alpha - \beta| = \frac{\sqrt{D}}{|a|}.$$

(c) **A quadratic equation whose roots are α & β is $(x - \alpha)(x - \beta) = 0$ i.e.**

$$x^2 - (\alpha + \beta)x + \alpha\beta = 0 \quad \text{i.e.}$$

$$x^2 - (\text{sum of roots})x + \text{product of roots} = 0.$$

Note.. 

$$y = (ax^2 + bx + c) \equiv a(x - \alpha)(x - \beta)$$

$$= a \left(x + \frac{b}{2a} \right)^2 - \frac{D}{4a}$$

3. NATURE OF ROOTS

(a) **Consider the quadratic equation $ax^2 + bx + c = 0$ where $a, b, c \in \mathbb{R}$ & $a \neq 0$ then;**

(i) $D > 0 \Leftrightarrow$ roots are real & distinct (unequal).

(ii) $D = 0 \Leftrightarrow$ roots are real & coincident (equal).

(iii) $D < 0 \Leftrightarrow$ roots are imaginary.

(iv) If $p + iq$ is one root of a quadratic equation, then the other must be the conjugate $p - iq$ & vice versa. ($p, q \in \mathbb{R}$ & $i = \sqrt{-1}$).

(b) **Consider the quadratic equation $ax^2 + bx + c = 0$ where $a, b, c \in \mathbb{Q}$ & $a \neq 0$ then;**

(i) If $D > 0$ & is a perfect square, then roots are rational & unequal.

(ii) If $\alpha = p + \sqrt{q}$ is one root in this case, (where p is rational & \sqrt{q} is a surd) then the other root must be the conjugate of it i.e. $\beta = p - \sqrt{q}$ & vice versa.

Note.. 

Remember that a quadratic equation cannot have three different roots & if it has, it becomes an identity.

4. GRAPH OF QUADRATIC EXPRESSION

Consider the quadratic expression, $y = ax^2 + bx + c$, $a \neq 0$ & $a, b, c \in \mathbb{R}$ then ;

- (i) The graph between x, y is always a parabola. If $a > 0$ then the shape of the parabola is concave upwards & if $a < 0$ then the shape of the parabola is concave downwards.
- (ii) $y > 0 \forall x \in \mathbb{R}$, only if $a > 0$ & $D < 0$
- (iii) $y < 0 \forall x \in \mathbb{R}$, only if $a < 0$ & $D < 0$

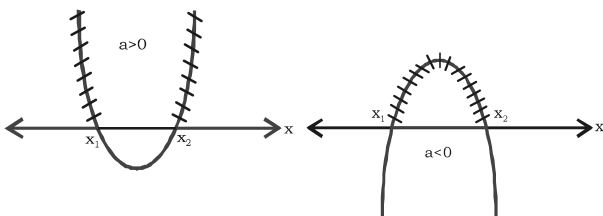
5. SOLUTION OF QUADRATIC INEQUALITIES

$ax^2 + bx + c > 0$ ($a \neq 0$).

- (i) If $D > 0$, then the equation $ax^2 + bx + c = 0$ has two different roots ($x_1 < x_2$).

Then $a > 0 \Rightarrow x \in (-\infty, x_1) \cup (x_2, \infty)$

$a < 0 \Rightarrow x \in (x_1, x_2)$



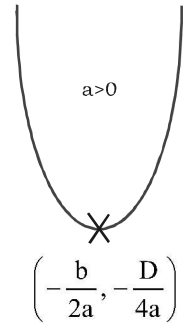
- (ii) Inequalities of the form $\frac{P(x)}{Q(x)} \geq 0$ can be quickly solved using the method of intervals (wavy curve).

6. MAX. & MIN. VALUE OF QUADRATIC EXPRESSION

Maximum & Minimum Value of $y = ax^2 + bx + c$ occurs at $x = -(b/2a)$ according as :

For $a > 0$, we have :

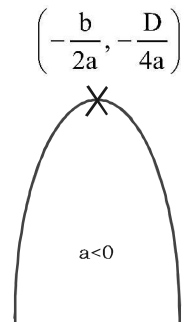
$y \in \left[\frac{4ac - b^2}{4a}, \infty \right)$



$y_{\min} = \frac{-D}{4a}$ at $x = \frac{-b}{2a}$, and $y_{\max} \rightarrow \infty$

For $a < 0$, we have :

$y \in \left(-\infty, \frac{4ac - b^2}{4a} \right]$



$y_{\max} = \frac{-D}{4a}$ at $x = \frac{-b}{2a}$, and $y_{\min} \rightarrow -\infty$

7. THEORY OF EQUATIONS

If $\alpha_1, \alpha_2, \alpha_3, \dots, \alpha_n$ are the roots of the n^{th} degree polynomial equation :

$$f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \dots + a_{n-1}x + a_n = 0$$

where a_0, a_1, \dots, a_n are all real & $a_0 \neq 0$,

Then,

$$\sum \alpha_1 = -\frac{a_1}{a_0};$$

$$\sum \alpha_1 \alpha_2 = \frac{a_2}{a_0};$$

$$\sum \alpha_1 \alpha_2 \alpha_3 = -\frac{a_3}{a_0};$$

.....

$$\alpha_1 \alpha_2 \alpha_3 \dots \alpha_n = (-1)^n \frac{a_n}{a_0}$$

8. LOCATION OF ROOTS

Let $f(x) = ax^2 + bx + c$, where $a > 0$ & $a, b, c \in R$.

- (i) Conditions for both the roots of $f(x) = 0$ to be greater than a specified number 'k' are :
 $D \geq 0$ & $f(k) > 0$ & $(-b/2a) > k$.
- (ii) Conditions for both roots of $f(x) = 0$ to lie on either side of the number 'k' (in other words the number 'k' lies between the roots of $f(x) = 0$ is :
 $a f(k) < 0$.
- (iii) Conditions for exactly one root of $f(x) = 0$ to lie in the interval (k_1, k_2) i.e. $k_1 < x < k_2$ are :
 $D > 0$ & $f(k_1) \cdot f(k_2) < 0$.
- (iv) Conditions that both roots of $f(x) = 0$ to be confined between the numbers k_1 & k_2 are $(k_1 < k_2)$:
 $D \geq 0$ & $f(k_1) > 0$ & $f(k_2) > 0$ & $k_1 < (-b/2a) < k_2$.



Remainder Theorem : If $f(x)$ is a polynomial, then $f(h)$ is the remainder when $f(x)$ is divided by $x - h$.

Factor theorem : If $x = h$ is a root of equation $f(x) = 0$, then $x-h$ is a factor of $f(x)$ and conversely.

9. MAX. & MIN. VALUES OF RATIONAL EXPRESSION

Here we shall find the values attained by a rational

expression of the form $\frac{a_1x^2 + b_1x + c_1}{a_2x^2 + b_2x + c_2}$ for real values

of x .

Example No. 4 will make the method clear.

10. COMMON ROOTS

(a) Only One Common Root

Let α be the common root of $ax^2 + bx + c = 0$ & $a'x^2 + b'x + c' = 0$, such that $a, a' \neq 0$ and $a' \neq a$.

Then, the condition for one common root is :

$$(ca' - c'a)^2 = (ab' - a'b)(bc' - b'c).$$

(b) Two Common Roots

Let α, β be the two common roots of

$$ax^2 + bx + c = 0 \text{ & } a'x^2 + b'x + c' = 0,$$

such that $a, a' \neq 0$.

Then, the condition for two common roots is :

$$\frac{a}{a'} = \frac{b}{b'} = \frac{c}{c'}$$

11. RESOLUTION INTO TWO LINEAR FACTORS

The condition that a quadratic function

$$f(x, y) = ax^2 + 2hxy + by^2 + 2gx + 2fy + c$$

may be resolved into two linear factors is that ;

$$abc + 2fgh - af^2 - bg^2 - ch^2 = 0$$

OR
$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$$

12. FORMATION OF A POLYNOMIAL EQUATION

If $\alpha_1, \alpha_2, \alpha_3, \dots, \alpha_n$ are the roots of the n^{th} degree polynomial equation, then the equation is

$$x^n - S_1x^{n-1} + S_2x^{n-2} + S_3x^{n-3} + \dots + (-1)^n S_n = 0$$

where S_k denotes the sum of the products of roots taken k at a time.

Particular Cases

(a) **Quadratic Equation** if α, β be the roots the quadratic equation, then the equation is :

$$x^2 - S_1x + S_2 = 0 \quad \text{i.e.} \quad x^2 - (\alpha + \beta)x + \alpha\beta = 0$$

(b) **Cubic Equation** if α, β, γ be the roots the cubic equation, then the equation is :

$$x^3 - S_1x^2 + S_2x - S_3 = 0 \quad \text{i.e.}$$

$$x^3 - (\alpha + \beta + \gamma)x^2 + (\alpha\beta + \beta\gamma + \gamma\alpha)x - \alpha\beta\gamma = 0$$

(i) If α is a root of equation $f(x) = 0$, the polynomial $f(x)$ is exactly divisible by $(x - \alpha)$. In other words, $(x - \alpha)$ is a factor of $f(x)$ and conversely.

(ii) Every equation of n^{th} degree ($n \geq 1$) has exactly n roots & if the equation has more than n roots, it is an identity.

(iii) If there be any two real numbers 'a' & 'b' such that $f(a)$ & $f(b)$ are of opposite signs, then $f(x) = 0$ must have atleast one real root between 'a' and 'b'.

(iv) Every equation $f(x) = 0$ of degree odd has atleast one real root of a sign opposite to that of its last term.

13. TRANSFORMATION OF EQUATIONS

(i) To obtain an equation whose roots are reciprocals of the roots of a given equation, it is obtained by replacing x by $1/x$ in the given equation

(ii) Transformation of an equation to another equation whose roots are negative of the roots of a given equation—replace x by $-x$.

(iii) Transformation of an equation to another equation whose roots are square of the roots of a given equation—replace x by \sqrt{x} .

(iv) Transformation of an equation to another equation whose roots are cubes of the roots of a given equation—replace x by $x^{1/3}$.